

# Coupling of HDG with a double-layer potential BEM

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## Abstract

In this paper we propose and analyze a new coupling procedure for the Hybridizable Discontinuous Galerkin Method with Galerkin Boundary Element Methods based on a double layer potential representation of the exterior component of the solution of a transmission problem. We show a discrete uniform coercivity estimate for the non-symmetric bilinear form and prove optimal convergence estimates for all the variables, as well as superconvergence for some of the discrete fields. Some numerical experiments support the theoretical findings.

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## 1 Introduction

In this paper we propose and analyze a new coupling procedure for the Hybridizable Discontinuous Galerkin Method (HDG) [6] and a Galerkin Boundary Element procedure based on a double layer potential representation. The model problem is a transmission problem in free space, coupling a linear diffusion equation with variable diffusivity ( $\operatorname{div} \kappa \nabla u = f$ ) in a polygonal domain, with an exterior Laplace equation. The transmission conditions are given by imposing the value of the difference of the unknown and its flux on the interface between the interior and the exterior domains.

Much has been written on the advantages and disadvantages of the many Discontinuous Galerkin schemes. In support of this piece of work, let us briefly hint at some of the features that make the HDG –originally a derivation of the Locally Discontinuous Galerkin (LDG) method by Bernardo Cockburn and his collaborators– a family of interest. First of all, HDG uses polynomials of the same degree  $k \geq 0$  for discretization of all the variables –in the case of diffusion problems, a scalar unknown  $u$ , the vector valued flux  $\mathbf{q} = -\kappa \nabla u$  and a scalar unknown on the skeleton of the triangulation–, attaining *optimal*

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order  $\mathcal{O}(h^{k+1})$  in the approximation of all of them. This feature compares well with Mixed Finite Element Methods, of which HDG is a natural modification: HDG can be considered as a variant of the Raviart-Thomas and Brezzi-Douglas-Marini Mixed Elements, implemented with Lagrange multipliers on the interelement faces [1], eliminating the degrees of freedom used for stabilization and using a stabilization (not penalization) parameter to perform the same task. Similar to the Lagrange multiplier implementation of mixed methods by Arnold and Brezzi [1], HDG is implemented by *hybridization*, reducing the unknowns to those living on interelement faces. This reduces considerably the number of degrees of freedom and makes the method competitive with respect to other families of Finite Element Methods, while still being a mixed method, that approximates several fields simultaneously. Another interesting feature of HDG is the fact that for polynomial degrees  $k \geq 1$ , the two unknowns related to the scalar field (the respective approximations inside the element and on the skeleton) *superconverge* at rate  $\mathcal{O}(h^{k+2})$ . This allows for the application of standard postprocessing techniques [7] that can be traced back to Stenberg [23].

Among many of its good properties (for the limited set of equations where it is usable), the Boundary Element Method provides a reliable form of constructing high order absorbing boundary conditions with complete flexibility on the geometric structure of the domain (it does not require the domain to be convex or even connected). It is therefore natural to test and study the possibility of using BEM as a way of generating a coupled discretization scheme with HDG for transmission problems. The coupling of DG and BEM started with the study of LDG-BEM schemes [14], extended to other DG methods of the Interior Penalty (IP) family [11]. All of these methods, plus several new ones, were presented as particular cases of a methodology for creating *symmetric couplings* of DG and BEM in [10]. One of the methods in this latter paper, using HDG and BEM, was recently analyzed in [8]. As opposed to symmetric couplings with BEM, that need two integral equations (and thus four integral operators) to reach a stable formulation, non-symmetric couplings require only one integral equation (and two integral operators). From this point of view, they provide simpler (and more natural) forms of coupling BEM with field methods (FEM, Mixed FEM or DG). The non-symmetric coupling of Mixed FEM and BEM is recent [17] and expands ideas used for analyzing the simple coupling of FEM-BEM [21]. The presentation and testing of non-symmetric coupling of DG methods of the IP family with BEM [19], led to its analysis [15], providing the first rigorous proof of convergence of this kind of schemes. Note that, unlike in the case of symmetric couplings, where the stability analysis boils down to an energy argument, coercivity properties in non-symmetric couplings pose a serious analytic challenge. The present paper makes a contribution in that direction, showing how to develop a coercivity analysis for non-symmetric coupling of HDG-BEM, and pointing out some interesting facts about the size of the diffusion parameter in the interior domain.

The paper is structured as follows. In Section 2 we present the method. Sections 3 to 5 cover the analysis in progressive steps. Section 6 discusses several related extensions and possible modifications of the method. In Section 7 we include some numerical experiments. A collection of known results concerning transmission problems, potentials and integral operators, is given in Appendix A for ease of reference.

**Regarding notation.** Basic theory of Sobolev spaces is assumed throughout. Norms of  $L^2(\mathcal{O})$  type will be subscripted with the integration domain  $\|\cdot\|_{\mathcal{O}}$ . All other norms will be indexed with the name of the space.

## 2 Formulation and discretization

Let  $\Omega \subset \mathbb{R}^d$  be a polygonal domain when  $d = 2$  or a Lipschitz polyhedral domain when  $d = 3$ , with boundary  $\Gamma$ . For simplicity we will assume that  $\Gamma$  is connected. (Connectedness plays a minor role in the amount of energy-free solutions of the transmission problem.) Let  $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega}$  be then the domain exterior to  $\Gamma$ . The outward pointing unit normal vector field on  $\Gamma$  is denoted  $\mathbf{n}$ .

### 2.1 Statement of the problem

We consider a scalar positive diffusion coefficient  $\kappa \in L^\infty(\Omega)$  such that  $\kappa^{-1} \in L^\infty(\Omega)$ . We are interested in the following transmission problem: an elliptic second order diffusion equation in  $\Omega$ , written as a first order system,

$$\mathbf{q} + \kappa \nabla u = \mathbf{0} \quad \text{and} \quad \operatorname{div} \mathbf{q} = f \quad \text{in } \Omega, \quad (2.1a)$$

the Laplace equation in the exterior domain

$$\Delta u_+ = 0 \quad \text{in } \Omega_+ \quad \text{and} \quad u = o(1) \quad \text{at infinity}, \quad (2.1b)$$

coupled through two transmission conditions on the common boundary of these domains

$$u = u_+ + \beta_0 \quad \text{and} \quad -\mathbf{q} \cdot \mathbf{n} = \partial_{\mathbf{n}} u_+ + \beta_1 \quad \text{on } \Gamma. \quad (2.1c)$$

In principle, boundary values of  $u$  and  $u_+$  in (2.1c) are taken in the sense of traces for functions with local Sobolev  $H^1$  regularity. Similarly  $\mathbf{q} \cdot \mathbf{n}$  and  $\partial_{\mathbf{n}} u_+$  are defined in a weak form as elements of the space  $H^{-1/2}(\Gamma)$ . Basic regularity requirements for data is:  $f \in L^2(\Omega)$ ,  $\beta_0 \in H^{1/2}(\Gamma)$ , and  $\beta_1 \in L^2(\Gamma)$ . We note that  $\beta_1 \in H^{-1/2}(\Gamma)$  is valid data for the transmission problem, but not for the type of formulation we are going to use. Data functions are assumed to satisfy the following compatibility condition:

$$\int_{\Omega} f + \int_{\Gamma} \beta_1 = 0. \quad (2.2)$$

A more detailed discussion about this condition and solvability issues for the transmission problems is given in Section A.4. Here we just note the following:

- (a) When  $d = 2$ , condition (2.2) is necessary and sufficient for the transmission problem (2.1) to have a solution. In this case  $u_+ = \mathcal{O}(r^{-1})$  as  $r = |\mathbf{x}| \rightarrow \infty$ .
- (b) When  $d = 3$ , problem (2.1) has a unique solution even if (2.2) fails to hold. If (2.2) is satisfied, the solution decays as  $u_+ = \mathcal{O}(r^{-2})$  at infinity. Proposition A.5(b) shows the elementary modification of data that is needed to have condition (2.2) satisfied, while still providing a solution of the original problem.

## 2.2 Coupling with a double layer potential

Condition (2.2) allows us to write  $u_+$  as a double layer potential

$$u_+ = \mathcal{D}\varphi, \quad \varphi \in H_0^{1/2}(\Gamma). \quad (2.3)$$

(See (A.2) and (A.6) for the definitions and Proposition A.2 for the representation result.) The exterior trace and normal derivative of the double layer potential can be written in terms of boundary integral operators, as explained in Section A.3. These operators are introduced in (A.12)-(A.14). They allow us to write the transmission conditions (2.1c) in the equivalent form

$$u = \frac{1}{2}\varphi + \mathcal{K}\varphi + \beta_0 \quad \text{and} \quad \mathbf{q} \cdot \mathbf{n} = \mathcal{W}\varphi - \beta_1 \quad \text{on } \Gamma. \quad (2.4)$$

Note that

$$\int_{\Gamma} \varphi = 0 \quad \text{and} \quad \mathcal{W}\varphi = \mathbf{q} \cdot \mathbf{n} + \beta_1 \quad \text{on } \Gamma \quad (2.5)$$

imply

$$\omega(\varphi, \phi) := \langle \mathcal{W}\varphi, \phi \rangle_{\Gamma} + \int_{\Gamma} \varphi \int_{\Gamma} \phi = \langle \mathbf{q} \cdot \mathbf{n} + \beta_1, \phi \rangle_{\Gamma} \quad \forall \phi \in H^{1/2}(\Gamma). \quad (2.6)$$

At this moment, it is convenient to collect all the equations that make up the formulation we are going to discretize:

$$\kappa^{-1}\mathbf{q} + \nabla u = 0 \quad \text{in } \Omega, \quad (2.7a)$$

$$\operatorname{div} \mathbf{q} = f \quad \text{in } \Omega, \quad (2.7b)$$

$$u - (\frac{1}{2}\varphi + \mathcal{K}\varphi) = \beta_0 \quad \text{on } \Gamma, \quad (2.7c)$$

$$-\langle \mathbf{q} \cdot \mathbf{n}, \phi \rangle_{\Gamma} + \omega(\varphi, \phi) = \langle \beta_1, \phi \rangle_{\Gamma} \quad \forall \phi \in H^{1/2}(\Gamma). \quad (2.7d)$$

The exterior field is reconstructed as a double layer potential  $u_+ = \mathcal{D}\varphi$ . The next lemma ensures that the integral boundary condition and the normalization condition (2.5) for the density  $\varphi$  are adequately encoded in the system (2.7).

**Lemma 2.1.** *If  $(\mathbf{q}, u, \varphi)$  is a solution to (2.7) and the compatibility condition (2.2) is satisfied, then  $\int_{\Gamma} \varphi = 0$ .*

*Proof.* Integrating (2.7b) over  $\Omega$  and testing (2.7d) with  $\phi = 1$ , it follows that

$$\int_{\Omega} f = \langle \mathbf{q} \cdot \mathbf{n}, 1 \rangle_{\Gamma} = \omega(\varphi, 1) - \langle \beta_1, 1 \rangle_{\Gamma} = |\Gamma| \int_{\Gamma} \varphi - \int_{\Gamma} \beta_1,$$

because of Proposition A.3. This proves the result.  $\square$

## 2.3 Discretization with HDG and Galerkin BEM

We start the discretization process by describing the discrete geometric elements. The domain  $\Omega$  is divided into triangles ( $d = 2$ ) or tetrahedra ( $d = 3$ ) with the usual conditions for conforming finite element meshes. A general element will be denoted  $K$ ,  $\mathcal{T}_h$  will be the set of all elements and  $h := \max_{K \in \mathcal{T}_h} h_K$  the maximum diameter of elements of the triangulation. The triangulation is assumed to be shape-regular. Extension of the forthcoming results to non-conforming meshes is relatively simple while not straightforward, requiring the use of some finely tuned results about HDG on general grids [3], [4].

We assume the existence of a triangulation  $\Gamma_h$  of the boundary  $\Gamma$ , formed by line segments ( $d = 2$ ) or triangles ( $d = 3$ ). We write  $h_\Gamma := \max_{e \in \Gamma_h} h_e$ . From the point of view of analysis, it is immaterial whether this grid is related to  $\mathcal{T}_h$  or not. (Note that for the experiments we will only use meshes where  $\Gamma_h$  is the trace of  $\mathcal{T}_h$  on  $\Gamma$ , which makes the implementation considerably simpler.) At the time of stating and proving the convergence theorems we will add some technical restrictions relating the mesh-sizes of the two grids.

The set of edges ( $d = 2$ ) or faces ( $d = 3$ ) of all elements of the triangulation will be denoted  $\mathcal{E}_h$  and  $\partial\mathcal{T}_h = \cup_{e \in \mathcal{E}_h} e$  will denote the skeleton of the triangulation. Volumetric integration will be denoted with parentheses

$$(u, v)_K := \int_K u v, \quad (\mathbf{p}, \mathbf{q})_K := \int_K \mathbf{p} \cdot \mathbf{q}, \quad (u, v)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (u, v)_K,$$

while integration on boundaries will be denoted with angled brackets

$$\langle u, v \rangle_{\partial K} := \int_{\partial K} u v, \quad \langle u, v \rangle_{\partial\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle u, v \rangle_{\partial K}, \quad \langle u, v \rangle_{\partial\mathcal{T}_h \setminus \Gamma} := \sum_{K \in \mathcal{T}_h} \langle u, v \rangle_{\partial K \setminus \Gamma}.$$

The unit normal outward pointing vector field on  $\partial K$  will be denoted  $\mathbf{n}_{\partial K}$ , or simply  $\mathbf{n}$ , when there is no doubt on what the element is.

Local discrete spaces will be composed of polynomials. The set  $\mathcal{P}_k(K)$  contains all  $d$ -variate polynomials of degree less than or equal to  $k$  and  $\mathcal{P}_k(K) := (\mathcal{P}_k(K))^d$ . If  $e \in \mathcal{E}_h$  or  $e \in \Gamma_h$ ,  $\mathcal{P}_k(e)$  is the set of polynomials of degree less than or equal to  $k$  defined on  $e$ , i.e., the space of  $(d - 1)$ -variate polynomials on local tangential coordinates. Four global spaces will be used for discretization:

$$\mathbf{V}_h := \{\mathbf{v} : \Omega \rightarrow \mathbb{R}^d : \mathbf{v}|_K \in \mathcal{P}_k(K) \quad \forall K \in \mathcal{T}_h\}, \quad (2.8a)$$

$$W_h := \{w : \Omega \rightarrow \mathbb{R} : w|_K \in \mathcal{P}_k(K) \quad \forall K \in \mathcal{T}_h\}, \quad (2.8b)$$

$$M_h := \{\hat{v} : \partial\mathcal{T}_h \rightarrow \mathbb{R} : \hat{v}|_e \in \mathcal{P}_k(e) \quad \forall e \in \mathcal{E}_h\}, \quad (2.8c)$$

$$Y_h := \{\phi : \Gamma \rightarrow \mathbb{R} : \phi \in \mathcal{C}(\Gamma), \quad \phi|_e \in \mathcal{P}_{k+1}(e) \quad \forall e \in \Gamma_h\}. \quad (2.8d)$$

The discretization method consists of using the Hybridizable Discontinuous Galerkin method (see [6], [7]) for equations (2.7a)-(2.7b)-(2.7c) (this part of the system can be understood as an interior Dirichlet problem) and conforming Galerkin (Boundary Element) Method for equation (2.7d) (this part is considered as a hypersingular integral equation

for an exterior Neumann problem). We thus look for  $\mathbf{q}_h \in \mathbf{V}_h$ ,  $u_h \in W_h$ ,  $\widehat{u}_h \in M_h$ , and  $\varphi_h \in Y_h$  satisfying

$$(\kappa^{-1} \mathbf{q}_h, \mathbf{r})_{\mathcal{T}_h} - (u_h, \operatorname{div} \mathbf{r})_{\mathcal{T}_h} + \langle \widehat{u}_h, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall \mathbf{r} \in \mathbf{V}_h, \quad (2.9a)$$

$$-(\mathbf{q}_h, \nabla w)_{\mathcal{T}_h} + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial \mathcal{T}_h} = (f, w)_{\mathcal{T}_h} \quad \forall w \in W_h, \quad (2.9b)$$

$$-\langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, \widehat{v} \rangle_{\partial \mathcal{T}_h \setminus \Gamma} = 0 \quad \forall \widehat{v} \in M_h, \quad (2.9c)$$

$$\langle \widehat{u}_h, \widehat{v} \rangle_{\Gamma} - \langle \tfrac{1}{2} \varphi_h + \mathcal{K} \varphi_h, \widehat{v} \rangle_{\Gamma} = \langle \beta_0, \widehat{v} \rangle_{\Gamma} \quad \forall \widehat{v} \in M_h, \quad (2.9d)$$

$$-\langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, \phi \rangle_{\Gamma} + \omega(\varphi_h, \phi) = \langle \beta_1, \phi \rangle_{\Gamma} \quad \forall \phi \in Y_h. \quad (2.9e)$$

Here  $\widehat{\mathbf{q}}_h$  is defined on the boundaries of the elements using the expression

$$\widehat{\mathbf{q}}_h := \mathbf{q}_h|_K + \tau_{\partial K}(u_h|_K - \widehat{u}_h) \mathbf{n}_{\partial K} \quad \text{on } \partial K, \quad (2.9f)$$

where  $\tau_{\partial K} : \partial K \rightarrow \mathbb{R}$  is a *non-negative* stabilization function that is constant on each edge/face of  $\partial K$  and such that it is strictly positive on at least one edge/face of each triangle. The stabilization function  $\tau$  can be double-valued on interelement edges/faces and  $\widehat{\mathbf{q}}_h$  is in principle multiple valued, but its normal component is made to be single valued through equation (2.9c). Note also that equations (2.9c)-(2.9d) are tested with the same space, but because of the integration domain, they produce together as many equations as the dimension of  $M_h$ . Using the local solvers for the HDG method [6], the system (2.9) can be reduced to a system with  $(\widehat{u}_h, \varphi_h) \in M_h \times Y_h$  as the only unknowns. A discrete counterpart of Lemma 2.1 holds.

**Lemma 2.2.** *If  $(\mathbf{q}_h, u_h, \widehat{u}_h, \varphi_h)$  solves (2.9) and the compatibility condition (2.2) holds, then  $\int_{\Gamma} \varphi_h = 0$ .*

*Proof.* Testing equation (2.9b) with  $w = 1$  and equation (2.9e) and using Proposition A.3, the result follows.  $\square$

The next sections deal with the analysis of the method: we first show the basic discrete coercivity arguments (Section 3), proceed to proving energy estimates based on the HDG projection (Section 4) and finally use duality arguments to analyze convergence (and superconvergence) of some of the fields (Section 5).

### 3 Solvability of the discrete system

The aim of this section is to show that, under some conditions on  $\kappa$ , the system (2.9) has a unique solution for  $h$  small enough. A relevant hypothesis will be the following:

$$\kappa_{\max} := \|\kappa\|_{L^\infty(\Omega)} < 4. \quad (3.1)$$

This hypothesis is related to coercivity of the underlying mixed formulation. We will discuss this hypothesis (and compare it with similar bounds for other coupled BEM-FEM schemes) in Section 7, where we will also make some experiments related to it. For some of the forthcoming arguments, we will use the piecewise constant function  $\mathfrak{h} : \partial \mathcal{T}_h \rightarrow \mathbb{R}$  given by  $\mathfrak{h}|_e := h_e$ . We will pay attention to the quantities

$$\tau_{\max} := \max_{K \in \mathcal{T}_h} \|\tau\|_{L^\infty(\partial K)} =: \|\tau\|_{L^\infty(\partial \mathcal{T}_h)} \quad \text{and} \quad \|\mathfrak{h}\tau\|_{L^\infty(\partial \mathcal{T}_h)},$$

noting that because of the shape regularity of the grid, we can bound  $h_K \|\tau\|_{L^\infty(\partial K)} \leq C \|\mathfrak{h}\tau\|_{L^\infty(\partial\mathcal{T}_h)}$  for all  $K$ . Three more polynomial spaces will appear in our arguments:

$$\begin{aligned}\mathcal{P}_k^\perp(K) &:= \{u \in \mathcal{P}_k(K) : (u, w)_K = 0 \quad \forall w \in \mathcal{P}_{k-1}(K)\}, \\ \mathcal{P}_k^\perp(K) &:= (\mathcal{P}_k^\perp(K))^d = \{\mathbf{q} \in \mathcal{P}_k(K) : (\mathbf{q}, \mathbf{r})_K = 0 \quad \forall \mathbf{r} \in \mathcal{P}_{k-1}(K)\}, \\ \mathcal{R}_k(\partial K) &:= \{q : \partial K \rightarrow \mathbb{R} : q|_e \in \mathcal{P}_k(e) \quad \forall e \in \mathcal{E}(K)\},\end{aligned}$$

where in the last space, we have denoted  $\mathcal{E}(K) := \{e \in \mathcal{E}_h : e \subset \partial K\}$ .

**Lemma 3.1.** (a) *If  $q \in \mathcal{P}_k^\perp(K)$  and  $q = 0$  on  $e \in \mathcal{E}(K)$ , then  $q = 0$ .*

(b) *The following decomposition is orthogonal in  $L^2(\partial K)$ :*

$$\mathcal{R}_k(\partial K) = \{\mathbf{v}|_{\partial K} \cdot \mathbf{n} : \mathbf{v} \in \mathcal{P}_k^\perp(K)\} \oplus \{q|_{\partial K} : q \in \mathcal{P}_k^\perp(K)\}.$$

*Proof.* Part (a) is straightforward. Part (b) is Lemma 4.1 of [9].  $\square$

**Lemma 3.2.** *Let  $P_k : L^2(\Omega) \rightarrow W_h$  be the orthogonal projection onto  $W_h$ . Then*

$$\|u - P_k u\|_{\partial K} \leq C h_K^{1/2} \|\nabla u\|_K \quad \forall K \in \mathcal{T}_h, \forall u \in H^1(\Omega).$$

*Proof.* Using the Bramble-Hilbert lemma (or a generalized Poincaré inequality), the trace theorem and an argument about finite dimensions, it is easy to prove that

$$\|\widehat{u} - \widehat{P}_k \widehat{u}\|_{\partial \widehat{K}} \leq C \|\nabla \widehat{u}\|_{\widehat{K}} \quad \forall \widehat{u} \in H^1(\widehat{K}),$$

where  $\widehat{K}$  is the reference element and  $\widehat{P}_k : L^2(\widehat{K}) \rightarrow \mathcal{P}_k(\widehat{K})$  is the orthogonal projector. The result follows then by a scaling argument.  $\square$

**Lemma 3.3** (A discrete coercivity estimate). *For discrete triples  $(\mathbf{q}_h, u_h, \widehat{u}_h) \in \mathbf{V}_h \times W_h \times M_h$  satisfying*

$$(\operatorname{div} \mathbf{q}_h, w)_{\mathcal{T}_h} + \langle \tau(u_h - \widehat{u}_h), w \rangle_{\partial\mathcal{T}_h} = 0 \quad \forall w \in W_h, \quad (3.2a)$$

$$\langle \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h), \widehat{v} \rangle_{\partial\mathcal{T}_h \setminus \Gamma} = 0 \quad \forall \widehat{v} \in M_h, \quad (3.2b)$$

and  $u_\star \in H^1(\Omega)$ , we consider the quadratic form:

$$Q := (\kappa^{-1} \mathbf{q}_h, \mathbf{q}_h)_{\mathcal{T}_h} + \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial\mathcal{T}_h} + \|\nabla u_\star\|_\Omega^2 - \langle u_\star, \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \rangle_\Gamma.$$

If (3.1) holds, then there exist  $C > 0$  and  $D > 0$  such that if  $\|\mathfrak{h}\tau\|_{L^\infty(\partial\mathcal{T}_h)} \leq D$ ,

$$Q \geq C \left( \|\mathbf{q}_h\|_\Omega^2 + \|\nabla u_\star\|_\Omega^2 + \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial\mathcal{T}_h} \right).$$

*Proof.* Condition (3.2b) is equivalent to the fact that the discrete normal fluxes  $\mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h)$  are single valued on internal faces/edges. Therefore

$$E_h := \langle u_\star, \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \rangle_\Gamma = \sum_{K \in \mathcal{T}_h} E_{\partial K}, \text{ where } E_{\partial K} := \langle u_\star, \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \rangle_{\partial K}.$$

If  $P_k : L^2(\Omega) \rightarrow W_h$  is the orthogonal projection onto  $W_h$  (as in Lemma 3.2), then

$$\begin{aligned} E_{\partial K} &= \langle \mathbf{q}_h \cdot \mathbf{n}, u_\star \rangle_{\partial K} + \langle \tau(u_h - \widehat{u}_h), P_k u_\star \rangle_{\partial K} + \langle \tau(u_h - \widehat{u}_h), u_\star - P_k u_\star \rangle_{\partial K} \\ &= \langle \mathbf{q}_h \cdot \mathbf{n}, u_\star \rangle_{\partial K} - \langle \operatorname{div} \mathbf{q}_h, P_k u_\star \rangle_K + \langle \tau(u_h - \widehat{u}_h), u_\star - P_k u_\star \rangle_{\partial K} \\ &= (\mathbf{q}_h, \nabla u_\star)_K + \langle \tau(u_h - \widehat{u}_h), u_\star - P_k u_\star \rangle_{\partial K}, \end{aligned}$$

where we have used (3.2a) and the fact that  $\operatorname{div} \mathbf{q}_h \in \mathcal{P}_{k-1}(K)$ . Therefore, by Lemma 3.2 and Young's inequality

$$\begin{aligned} |E_{\partial K}| &\leq \|\mathbf{q}_h\|_K \|\nabla u_\star\|_K + C(h_K \|\tau\|_{L^\infty(\partial K)})^{1/2} \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial K}^{1/2} \|\nabla u_\star\|_K \\ &\leq \delta^{-1} \|\mathbf{q}_h\|_K^2 + C^2 \|\mathfrak{h}\tau\|_{L^\infty(\partial \mathcal{T}_h)} \delta^{-1} \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial K} + \frac{1}{4}(\delta + \underline{\delta}) \|\nabla u_\star\|_K^2, \end{aligned}$$

for arbitrary  $\delta, \underline{\delta} > 0$ . Therefore

$$\begin{aligned} Q &= (\kappa^{-1} \mathbf{q}_h, \mathbf{q}_h) + \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial \mathcal{T}_h} + \|\nabla u_\star\|_\Omega^2 - E_h \\ &\geq (\kappa_{\max}^{-1} - \delta^{-1}) \|\mathbf{q}_h\|_\Omega^2 + (1 - C^2 \|\mathfrak{h}\tau\|_{L^\infty(\partial \mathcal{T}_h)} \delta^{-1}) \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial \mathcal{T}_h} \\ &\quad + (1 - \frac{1}{4}(\delta + \underline{\delta})) \|\nabla u_\star\|_\Omega^2 \\ &\geq \frac{4 - \kappa_{\max}}{\kappa_{\max}(4 + \kappa_{\max})} \|\mathbf{q}_h\|_\Omega^2 + \frac{3}{4} \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial \mathcal{T}_h} + \frac{4 - \kappa_{\max}}{16} \|\nabla u_\star\|_\Omega^2, \end{aligned}$$

where we have chosen

$$\delta := \frac{4 + \kappa_{\max}}{2} \quad \underline{\delta} := \frac{4 - \kappa_{\max}}{4}, \quad \|\mathfrak{h}\tau\|_{L^\infty(\partial \mathcal{T}_h)} \leq \frac{4 - \kappa_{\max}}{16C^2}.$$

This finishes the proof.  $\square$

**Proposition 3.4.** *If (3.1) holds, then there exists  $D > 0$  such that for all  $\|\mathfrak{h}\tau\|_{L^\infty(\partial \mathcal{T}_h)} \leq D$ , the system (2.9) is uniquely solvable. In particular, if  $\tau_{\max} \leq C$ , the system is uniquely solvable for  $h$  small enough.*

*Proof.* We only need to prove that the homogeneous system only admits the trivial solution. Let then  $(\mathbf{q}_h, u_h, \widehat{u}_h, \varphi_h)$  be a solution of (2.9) with zero right hand side. By Lemma 2.2 it follows that  $\omega(\varphi_h, \psi) = \langle \mathcal{W}\varphi_h, \psi \rangle_\Gamma$ . We now test equations (2.9) with  $\mathbf{q}_h$ ,  $u_h$ ,  $\widehat{u}_h$ ,  $-\mathbf{q}_h \cdot \mathbf{n} - \tau(u_h - \widehat{u}_h)$  and  $\varphi_h$  respectively, add the equations, and simplify the result to obtain

$$(\kappa^{-1} \mathbf{q}_h, \mathbf{q}_h)_{\mathcal{T}_h} + \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial \mathcal{T}_h} + \langle -\frac{1}{2} \varphi_h + \mathcal{K} \varphi_h, \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \rangle_\Gamma + \langle \mathcal{W}\varphi_h, \varphi_h \rangle_\Gamma = 0.$$

Let now  $u_\star := \mathcal{D}\varphi_h$ . Using then (A.13) and Proposition A.3, it follows that

$$(\kappa^{-1} \mathbf{q}_h, \mathbf{q}_h)_{\mathcal{T}_h} + \langle \tau(u_h - \widehat{u}_h), u_h - \widehat{u}_h \rangle_{\partial \mathcal{T}_h} + \langle \gamma^- u_\star, \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \rangle_\Gamma + \|\nabla u_\star\|_{\mathbb{R}^d \setminus \Gamma}^2 = 0,$$

where  $\gamma^- u_\star$  makes reference to the trace of  $u_\star|_\Omega$  on  $\Gamma$ . We are now in the hypotheses of Lemma 3.3, which implies that  $\mathbf{q}_h = \mathbf{0}$ ,  $\nabla u_\star = \mathbf{0}$ , and  $\tau(u_h - \widehat{u}_h) = 0$  on  $\partial K$  for all  $K$ . By Proposition A.1 and the fact that  $\int_\Gamma \varphi_h = 0$ , this implies that  $\varphi_h = 0$ .

Going back to (2.9a), it follows that

$$(\nabla u_h, \mathbf{r})_K + \langle u_h - \widehat{u}_h, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial K} = 0 \quad \forall \mathbf{r} \in \mathcal{P}_k(K) \quad \forall K \quad (3.3)$$



and therefore

$$\langle u_h - \widehat{u}_h, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial K} = 0 \quad \forall \mathbf{r} \in \mathcal{P}_k^\perp(K) \quad \forall K.$$

By Lemma 3.1(b), there exists  $q \in \mathcal{P}_k^\perp(K)$  such that  $u_h - \widehat{u}_h = q$  of  $\partial K$ . Since  $\tau(u_h - \widehat{u}_h) = 0$  and  $\tau \neq 0$  on at least one  $e \in \mathcal{E}(K)$ , it follows by Lemma 3.1(a) that  $q = 0$  and therefore  $u_h = \widehat{u}_h$  on  $\partial \mathcal{T}_h$ . Using this information in (3.3) and testing with  $\mathbf{r} = \nabla u_h$ , it follows that  $u_h$  is piecewise constant. At the same time,  $u_h = \widehat{u}_h$  on  $\partial \mathcal{T}_h$ , which implies that both  $u_h$  and  $\widehat{u}_h$  are constant. Testing (2.9d) with  $\widehat{v} = 1$  implies that this constant value has to vanish. This finishes the proof of uniqueness.  $\square$

## 4 Estimates by energy arguments

Convergence analysis of the coupled HDG-BEM scheme follows the main lines of the projection-based analysis of HDG methods [7] (see also [8]). We start by recalling the local HDG projection [7]. Given  $(\mathbf{q}, u)$ , we define  $(\Pi \mathbf{q}, \Pi u) \in \mathbf{V}_h \times W_h$  as the solution of the local problems

$$(\Pi \mathbf{q}, \mathbf{r})_K = (\mathbf{q}, \mathbf{r})_K \quad \forall \mathbf{r} \in \mathcal{P}_{k-1}(K) \quad (4.1a)$$

$$(\Pi u, w)_K = (u, w) \quad \forall w \in \mathcal{P}_{k-1}(K) \quad (4.1b)$$

$$\langle \Pi \mathbf{q} \cdot \mathbf{n} + \tau \Pi u, \widehat{v} \rangle_{\partial K} = \langle \mathbf{q} \cdot \mathbf{n} + \tau u, \widehat{v} \rangle_{\partial K} \quad \forall \widehat{v} \in \mathcal{R}_k(\partial K), \quad (4.1c)$$

for all  $K \in \mathcal{T}_h$ . We also consider the orthogonal projections  $P : \prod_{e \in \mathcal{E}_h} L^2(e) \rightarrow M_h$  and  $P_\Gamma : H^{1/2}(\Gamma) \rightarrow Y_h$ . The error analysis is carried out by comparing the discrete solution with the projection of the exact solution, i.e., in terms of the quantities:

$$\boldsymbol{\varepsilon}_h^q := \Pi \mathbf{q} - \mathbf{q}_h, \quad \varepsilon_h^u := \Pi u - u_h, \quad \widehat{\varepsilon}_h^u := Pu - \widehat{u}_h, \quad \varepsilon_h^\varphi := P_\Gamma \varphi - \varphi_h.$$

The error in the normal flux (recall (2.9f)) is

$$\widehat{\varepsilon}_h := \boldsymbol{\varepsilon}_h^q \cdot \mathbf{n} + \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u) = \widehat{\pi} - \widehat{\mathbf{q}}_h \cdot \mathbf{n}, \quad \text{where} \quad \widehat{\pi} := \Pi \mathbf{q} \cdot \mathbf{n} + \tau(\Pi u - Pu).$$

**Lemma 4.1.** *The following inequalities hold for all  $K$  and  $e \in \mathcal{E}(K)$ :*

$$h_e \|\widehat{\pi} - \mathbf{q} \cdot \mathbf{n}\|_e^2 \leq C \left( \|\mathbf{q} - \Pi \mathbf{q}\|_K^2 + h_K^2 \|\nabla(\mathbf{P}_k \mathbf{q} - \mathbf{q})\|_K^2 \right), \quad (4.2)$$

$$\|\widehat{\pi} - \mathbf{q} \cdot \mathbf{n}\|_e^2 \leq C h_K \|\nabla \mathbf{q}\|_K^2, \quad (4.3)$$

where  $\mathbf{P}_k \mathbf{q}$  is the best  $L^2(\Omega)^d$  approximation of  $\mathbf{q}$  in  $\mathbf{V}_h$ .

*Proof.* Note also that by definition of the projections

$$\langle \widehat{\pi}, \widehat{v} \rangle_{\partial K} = \langle \mathbf{q} \cdot \mathbf{n}, \widehat{v} \rangle_{\partial K} \quad \forall \widehat{v} \in \mathcal{R}_k(\partial K) \quad \forall K. \quad (4.4)$$

This shows that  $\widehat{\pi}|_e$  is the best  $L^2(e)$  approximation of  $\mathbf{q} \cdot \mathbf{n}_e$  on  $\mathbb{P}_k(e)$ . Using then a local trace inequality, we can easily bound for  $e \in \mathcal{E}(K)$

$$h_e \|\widehat{\pi} - \mathbf{q} \cdot \mathbf{n}\|_e^2 \leq h_e \|\mathbf{P}_k \mathbf{q} - \mathbf{q}\|_e^2 \leq C \left( \|\mathbf{P}_k \mathbf{q} - \mathbf{q}\|_K^2 + h_K^2 \|\nabla(\mathbf{P}_k \mathbf{q} - \mathbf{q})\|_K^2 \right). \quad (4.5)$$

The second inequality follows from a similar argument, comparing with  $\mathbf{P}_0 \mathbf{q}$  instead of  $\mathbf{P}_k \mathbf{q}$ .  $\square$

For simplicity, in some of the forthcoming arguments we will shorten  $\tilde{\mathcal{K}} := \frac{1}{2}\mathcal{I} + \mathcal{K}$ . The following three discrete functionals will be relevant in the sequel as well:

$$c_1(\mathbf{r}) := (\kappa^{-1}(\mathbf{\Pi}\mathbf{q} - \mathbf{q}), \mathbf{r})_{\mathcal{T}_h}, \quad (4.6a)$$

$$c_2(\hat{v}) := \langle \tilde{\mathcal{K}}(P_\Gamma \varphi - \varphi), \hat{v} \rangle_\Gamma, \quad (4.6b)$$

$$c_3(\phi) := -\langle \hat{\pi} - \mathbf{q} \cdot \mathbf{n}, \phi \rangle_\Gamma + \omega(P_\Gamma \varphi - \varphi, \phi). \quad (4.6c)$$

**Proposition 4.2** (Energy inequality). *If (3.1) holds, then there exists  $h_0$  such that for all  $h \leq h_0$ ,*

$$\|\boldsymbol{\varepsilon}_h^q\|_\Omega^2 + \langle \tau(\varepsilon_h^u - \hat{\varepsilon}_h^u), \varepsilon_h^u - \hat{\varepsilon}_h^u \rangle_{\partial\mathcal{T}_h} + \|\varepsilon_h^\varphi\|_{H^{1/2}(\Gamma)}^2 \leq C \left( c_1(\boldsymbol{\varepsilon}_h^q) + c_2(\hat{\varepsilon}_h) + c_3(\varepsilon_h^\varphi) \right). \quad (4.7)$$

*Proof.* By the definition of the projections and (4.4), it follows that

$$(\kappa^{-1}\mathbf{\Pi}\mathbf{q}, \mathbf{r})_{\mathcal{T}_h} - (\mathbf{\Pi}u, \operatorname{div} \mathbf{r})_{\mathcal{T}_h} + \langle Pu, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} = c_1(\mathbf{r}) \quad \forall \mathbf{r} \in \mathbf{V}_h, \quad (4.8a)$$

$$-(\mathbf{\Pi}\mathbf{q}, \nabla w)_{\mathcal{T}_h} + \langle \hat{\pi}, w \rangle_{\partial\mathcal{T}_h} = (f, w)_{\mathcal{T}_h} \quad \forall w \in W_h, \quad (4.8b)$$

$$-\langle \hat{\pi}, \hat{v} \rangle_{\partial\mathcal{T}_h \setminus \Gamma} = 0 \quad \forall \hat{v} \in M_h, \quad (4.8c)$$

$$\langle Pu, \hat{v} \rangle_\Gamma - \langle \tilde{\mathcal{K}}P_\Gamma \varphi, \hat{v} \rangle_\Gamma = \langle \beta_0, \hat{v} \rangle_\Gamma - c_2(\hat{v}) \quad \forall \hat{v} \in M_h, \quad (4.8d)$$

$$-\langle \hat{\pi}, \phi \rangle_\Gamma + \omega(P_\Gamma \varphi, \phi) = \langle \beta_1, \phi \rangle_\Gamma + c_3(\phi) \quad \forall \phi \in Y_h. \quad (4.8e)$$

Subtracting (2.9) from equations (4.8) we obtain the *error equations*:

$$(\kappa^{-1}\boldsymbol{\varepsilon}_h^q, \mathbf{r})_{\mathcal{T}_h} - (\varepsilon_h^u, \operatorname{div} \mathbf{r})_{\mathcal{T}_h} + \langle \hat{\varepsilon}_h^u, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} = c_1(\mathbf{r}) \quad \forall \mathbf{r} \in \mathbf{V}_h, \quad (4.9a)$$

$$(\operatorname{div} \boldsymbol{\varepsilon}_h^q, w)_{\mathcal{T}_h} + \langle \tau(\varepsilon_h^u - \hat{\varepsilon}_h^u), w \rangle_{\partial\mathcal{T}_h} = 0 \quad \forall w \in W_h, \quad (4.9b)$$

$$-\langle \hat{\varepsilon}_h, \hat{v} \rangle_{\partial\mathcal{T}_h \setminus \Gamma} = 0 \quad \forall \hat{v} \in M_h, \quad (4.9c)$$

$$\langle \hat{\varepsilon}_h^u, \hat{v} \rangle_\Gamma - \langle \tilde{\mathcal{K}}\varepsilon_h^\varphi, \hat{v} \rangle_\Gamma = -c_2(\hat{v}) \quad \forall \hat{v} \in M_h, \quad (4.9d)$$

$$-\langle \hat{\varepsilon}_h, \phi \rangle_\Gamma + \omega(\varepsilon_h^\varphi, \phi) = c_3(\phi) \quad \forall \phi \in Y_h. \quad (4.9e)$$

Testing these equations with  $\boldsymbol{\varepsilon}_h^q$ ,  $\varepsilon_h^u$ ,  $\hat{\varepsilon}_h^u$ ,  $-\hat{\varepsilon}_h$  and  $\varepsilon_h^\varphi$  respectively, adding them, and simplifying, we obtain

$$\begin{aligned} (\kappa^{-1}\boldsymbol{\varepsilon}_h^q, \boldsymbol{\varepsilon}_h^q)_{\mathcal{T}_h} + \langle \tau(\varepsilon_h^u - \hat{\varepsilon}_h^u), \varepsilon_h^u - \hat{\varepsilon}_h^u \rangle_{\partial\mathcal{T}_h} + \langle \mathcal{K}\varepsilon_h^\varphi - \frac{1}{2}\varepsilon_h^\varphi, \hat{\varepsilon}_h \rangle_\Gamma + \omega(\varepsilon_h^\varphi, \varepsilon_h^\varphi) \\ = c_1(\boldsymbol{\varepsilon}_h^q) + c_2(\hat{\varepsilon}_h) + c_3(\varepsilon_h^\varphi). \end{aligned} \quad (4.10)$$

Let now  $\varepsilon_h^\star := \mathcal{D}\varepsilon_h^\varphi$  and note that Proposition A.3 and (A.13) imply that

$$\omega(\varepsilon_h^\varphi, \varepsilon_h^\varphi) = \|\nabla \varepsilon_h^\star\|_{\mathbb{R}^d \setminus \Gamma}^2 + \left| \int_\Gamma \varepsilon_h^\varphi \right|^2 \quad \text{and} \quad \mathcal{K}\varepsilon_h^\varphi - \frac{1}{2}\varepsilon_h^\varphi = -\gamma^- \varepsilon_h^\star.$$

Applying now the coercivity estimate of Lemma 3.3, it follows from (4.10) that

$$\|\boldsymbol{\varepsilon}_h^q\|_\Omega^2 + \langle \tau(\varepsilon_h^u - \hat{\varepsilon}_h^u), \varepsilon_h^u - \hat{\varepsilon}_h^u \rangle_{\partial\mathcal{T}_h} + \omega(\varepsilon_h^\varphi, \varepsilon_h^\varphi) \leq C \left( c_1(\boldsymbol{\varepsilon}_h^q) + c_2(\hat{\varepsilon}_h) + c_3(\varepsilon_h^\varphi) \right).$$

Since the bilinear form  $\omega$  is  $H^{1/2}(\Gamma)$ -coercive (Proposition A.3), the result follows.  $\square$

At this moment we have to introduce a technical hypothesis,

$$h_\Gamma \leq C \min\{h_K : K \in \mathcal{T}_h, \overline{K} \cap \Gamma \neq \emptyset\}, \quad \text{i.e.,} \quad \|\mathfrak{h}^{-1}\|_{L^\infty(\Gamma)} \leq Ch_\Gamma^{-1}, \quad (4.11)$$

meaning that the meshsize of  $\Gamma_h$  is controlled by the minimum local meshsize of  $\mathcal{T}_h$  on  $\Gamma$ . This hypothesis is satisfied, for instance, if  $\mathcal{T}_h$  is quasiuniform in a neighborhood of  $\Gamma$  and  $\Gamma_h$  is equal the restriction of  $\mathcal{T}_h$  to  $\Gamma$ , or is a refinement of this inherited mesh. We will also assume that

$$h_\Gamma \|\tau\|_{L^\infty(\Gamma)} \leq C. \quad (4.12)$$

**Theorem 4.3.** *Assume that (3.1), (4.11) and (4.12) hold. Then for  $\|\mathfrak{h}\tau\|_{L^\infty(\partial\mathcal{T}_h)}$  small enough*

$$\|\varepsilon_h^q\|_\Omega + \langle \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u), \varepsilon_h^u - \widehat{\varepsilon}_h^u \rangle_{\partial\mathcal{T}_h}^{1/2} + \|\varepsilon_h^\varphi\|_{H^{1/2}(\Gamma)} \leq C \left( \text{App}_h^q + \|P_\Gamma\varphi - \varphi\|_{H^{1/2}(\Gamma)} \right),$$

where

$$\text{App}_h^q := \|\mathbf{P}\mathbf{q} - \mathbf{q}\|_\Omega + \left( \sum_K h_K^2 \|\nabla(\mathbf{P}_K\mathbf{q} - \mathbf{q})\|_K^2 \right)^{1/2}.$$

*Proof.* We will derive the proof from Proposition 4.2 and some estimates related to the functionals defined in (4.6). First of all

$$|c_1(\varepsilon_h^q)| \leq C \|\mathbf{P}\mathbf{q} - \mathbf{q}\|_\Omega \|\varepsilon_h^q\|_\Omega. \quad (4.13)$$

Using Proposition A.4 and a scaling argument (note the use of (4.11) in the scaling argument), it follows that

$$\begin{aligned} |c_2(\widehat{\varepsilon}_h)| &\leq \|\widetilde{K}(P_\Gamma\varphi - \varphi)\|_\Gamma (\|\varepsilon_h^q \cdot \mathbf{n}\|_\Gamma + \|\tau\|_{L^\infty(\Gamma)}^{1/2} \langle \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u), \varepsilon_h^u - \widehat{\varepsilon}_h^u \rangle_\Gamma^{1/2}) \\ &\leq C \|P_\Gamma\varphi - \varphi\|_\Gamma \left( h_\Gamma^{-1/2} \|\varepsilon_h^q\|_\Omega + \|\tau\|_{L^\infty(\Gamma)}^{1/2} \langle \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u), \varepsilon_h^u - \widehat{\varepsilon}_h^u \rangle_\Gamma^{1/2} \right). \end{aligned} \quad (4.14)$$

An Aubin-Nitsche duality argument and properties of best approximation by piecewise polynomial functions shows that

$$\|P_\Gamma\varphi - \varphi\|_\Gamma \leq Ch_\Gamma^{1/2} \|P_\Gamma\varphi - \varphi\|_{H^{1/2}(\Gamma)}. \quad (4.15)$$

For the third functional, we use Proposition A.3 and (4.4)

$$|c_3(\varepsilon_h^\varphi)| \leq |\langle \widehat{\pi} - \mathbf{q} \cdot \mathbf{n}, \varepsilon_h^\varphi - P\varepsilon_h^\varphi \rangle_\Gamma| + C \|P_\Gamma\varphi - \varphi\|_{H^{1/2}(\Gamma)} \|\varepsilon_h^\varphi\|_{H^{1/2}(\Gamma)}, \quad (4.16)$$

where it is to be understood that the local projection  $P\varepsilon_h^\varphi$  is only applied (and needed) on  $\Gamma$ . A local scaling argument (cf [14, Theorem 3.2]) shows that

$$\|\mathfrak{h}^{-1/2}(\varepsilon_h^\varphi - P\varepsilon_h^\varphi)\|_\Gamma \leq C \|\varepsilon_h^\varphi\|_{H^{1/2}(\Gamma)}. \quad (4.17)$$

This inequality and Lemma 4.1 can then be used in (4.16) to prove that

$$|c_3(\varepsilon_h^\varphi)| \leq C \left( \text{App}_h^q + \|P_\Gamma\varphi - \varphi\|_{H^{1/2}(\Gamma)} \right) \|\varepsilon_h^\varphi\|_{H^{1/2}(\Gamma)} \quad (4.18)$$

The result is then a consequence of Proposition 4.2 and the bounds (4.13), (4.14), (4.15), and (4.18).  $\square$

**Corollary 4.4.** *In the hypotheses of Theorem 4.3,*

$$\|\mathfrak{h}^{1/2}(\mathbf{q} \cdot \mathbf{n} - \widehat{\mathbf{q}}_h \cdot \mathbf{n})\|_{\partial\mathcal{T}_h} \leq C(\text{App}_h^q + \|P_\Gamma\varphi - \varphi\|_{H^{1/2}(\Gamma)}).$$

*Proof.* Adding and subtracting  $\widehat{\pi}$ , we can bound

$$\begin{aligned} \|\mathfrak{h}^{1/2}(\mathbf{q} \cdot \mathbf{n} - \widehat{\mathbf{q}}_h \cdot \mathbf{n})\|_{\partial\mathcal{T}_h} &\leq \|\mathfrak{h}^{1/2}\widehat{\varepsilon}_h\|_{\partial\mathcal{T}_h} + \|\mathfrak{h}^{1/2}(\widehat{\pi} - \mathbf{q} \cdot \mathbf{n})\|_{\partial\mathcal{T}_h} \\ &\leq \|\mathfrak{h}^{1/2}\boldsymbol{\varepsilon}_h^q \cdot \mathbf{n}\|_{\partial\mathcal{T}_h} + \|\mathfrak{h}\tau\|_{L^\infty(\partial\mathcal{T}_h)} \langle \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u, \varepsilon_h^u - \widehat{\varepsilon}_h^u) \rangle_{\partial\mathcal{T}_h} + \|\mathfrak{h}^{1/2}(\widehat{\pi} - \mathbf{q} \cdot \mathbf{n})\|_{\partial\mathcal{T}_h}. \end{aligned}$$

A scaling argument (using the fact that  $\boldsymbol{\varepsilon}_h^q \in \mathbf{V}_h$ ), plus Lemma 4.1 and Theorem 4.3 finish the proof.  $\square$

Let  $\Gamma_1, \dots, \Gamma_L$  be the faces (with  $d = 3$ ) or edges (when  $d = 2$ ) of  $\Gamma$ . We consider the space  $X^m(\Gamma) := \prod_{\ell=1}^L H^m(\Gamma)$ , endowed with its product norm.

**Corollary 4.5.** *Assume that the hypotheses of Theorem 4.3 hold and that the exact solution of (2.7) satisfies:  $\mathbf{q} \in H^{k+1}(\Omega)^d$ ,  $u \in H^{k+1}(\Omega)$ ,  $\varphi \in X^{k+2}(\Gamma)$ . Then*

$$\begin{aligned} \|\mathbf{q} - \mathbf{q}_h\|_\Omega + \|\varphi - \varphi_h\|_{H^{1/2}(\Gamma)} \\ \leq Ch^{k+1}(|\mathbf{q}|_{H^{k+1}(\Omega)} + \delta_\tau |u|_{H^{k+1}(\Omega)}) + Ch_\Gamma^{k+3/2} \|\varphi\|_{X^{k+2}(\Gamma)}, \end{aligned}$$

where  $\delta_\tau := \max_K \|\tau\|_{L^\infty(\partial K \setminus e_K)}$  and  $e_K \in \mathcal{E}(K)$  is such that  $\tau_{e_K} = \|\tau\|_{L^\infty(\partial K)}$ .

*Proof.* This bound is a direct consequence of Theorem 4.3, using well-known estimates for the best approximation by piecewise polynomials and [7, Theorem 2.1].  $\square$

## 5 Estimates by duality arguments

Consider the problem

$$\kappa^{-1}\mathbf{d} + \nabla\Theta = 0 \quad \text{in } \Omega, \quad (5.1a)$$

$$\text{div } \mathbf{d} = \varepsilon_h^u \quad \text{in } \Omega, \quad (5.1b)$$

$$\Delta\omega = 0 \quad \text{in } \Omega_+, \quad (5.1c)$$

$$\omega = c_\infty\Phi(r) + \mathcal{O}(r^{-d+1}) \quad \text{at infinity}, \quad (5.1d)$$

$$\Theta = \omega \quad \text{on } \Gamma, \quad (5.1e)$$

$$\mathbf{d} \cdot \mathbf{n} = -\partial_{\mathbf{n}}\omega \quad \text{on } \Gamma. \quad (5.1f)$$

By Proposition A.2 and (5.1c)-(5.1f), we can represent  $\omega = \mathcal{D}\Theta + \mathcal{S}(\mathbf{d} \cdot \mathbf{n})$  in  $\Omega_+$ . Therefore, using the definition of the operator  $\mathcal{W}$ —see (A.12)—and Proposition A.4, we can write

$$-\langle \mathbf{d} \cdot \mathbf{n}, \tfrac{1}{2}\psi + \mathcal{K}\psi \rangle_\Gamma + \langle \mathcal{W}\psi, \Theta \rangle_\Gamma = 0 \quad \forall \psi \in H^{1/2}(\Gamma). \quad (5.2)$$

We will assume the following regularity estimate

$$\|\Theta\|_{H^2(\Omega)} + \|\mathbf{d}\|_{H^1(\Omega)} \leq C_{\text{reg}} \|\varepsilon_h^u\|_\Omega. \quad (5.3)$$

Since this is a transmission problem, this hypothesis is mainly related to the regularity of the diffusion coefficient  $\kappa$ . If  $\kappa$  is smooth and equal to one in a neighborhood of  $\Gamma$ , then (5.3) holds (see [8, Proposition 3.4] for a similar argument).

**Proposition 5.1.** *In the hypotheses of Theorem 4.3 and assuming the regularity estimate (5.3), it holds*

$$\|\varepsilon_h^u\|_\Omega \leq Ch^{\min\{k,1\}} \left( \text{App}_h^q + \|P_\Gamma \varphi - \varphi\|_{H^{1/2}(\Gamma)} \right). \quad (5.4)$$

*Proof.* The gist of the proof consists of testing the error equations (4.9) with  $\Pi \mathbf{d}$ ,  $-\Pi \Theta$ ,  $-P\Theta$ ,  $-(\Pi \mathbf{d} + \tau(\Pi \Theta - P\Theta))$ , and  $-P_\Gamma \Theta$  respectively, add them and manipulate the result. Let us first test the first error equation (4.9a) with  $\mathbf{r} = \Pi \mathbf{d}$ :

$$(\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \Pi \mathbf{d})_{\mathcal{T}_h} - (\varepsilon_h^u, \text{div } \Pi \mathbf{d})_{\mathcal{T}_h} + \langle \widehat{\varepsilon}_h^u, \Pi \mathbf{d} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0.$$

Using integration by parts twice, the first condition in the definition of the HDG projection (4.1a) and the fact that  $\text{div } \mathbf{d} = \varepsilon_h^u$ , we can write

$$(\text{div } \Pi \mathbf{d}, \varepsilon_h^u)_{\mathcal{T}_h} = -(\mathbf{d}, \nabla \varepsilon_h^u)_{\mathcal{T}_h} + \langle \varepsilon_h^u, \Pi \mathbf{d} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = \|\varepsilon_h^u\|_\Omega^2 + \langle \varepsilon_h^u, (\Pi \mathbf{d} - \mathbf{d}) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}.$$

Adding these two equations, it follows that

$$\begin{aligned} \|\varepsilon_h^u\|_\Omega^2 &= (\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \Pi \mathbf{d})_{\mathcal{T}_h} - \langle \varepsilon_h^u - \widehat{\varepsilon}_h^u, (\Pi \mathbf{d} - \mathbf{d}) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} + \langle \widehat{\varepsilon}_h^u, \mathbf{d} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &= (\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \Pi \mathbf{d})_{\mathcal{T}_h} + \langle \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u), \Pi \Theta - P\Theta \rangle_{\partial \mathcal{T}_h} + \langle \widehat{\varepsilon}_h^u, \mathbf{d} \cdot \mathbf{n} \rangle_\Gamma, \end{aligned} \quad (5.5)$$

where in the last equation we have applied (4.1c) and the fact that  $\widehat{\varepsilon}_h^u$  and  $\mathbf{d}$  are single valued on interelement faces. We next test (4.9b) with  $w = \Pi \Theta$  and apply (4.1b) and the fact that  $\nabla \Theta = -\kappa^{-1} \mathbf{d}$ :

$$\begin{aligned} 0 &= (\text{div } \boldsymbol{\varepsilon}_h^q, \Theta)_{\mathcal{T}_h} + \langle \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u), \Pi \Theta \rangle_{\partial \mathcal{T}_h} \\ &= (\kappa^{-1} \boldsymbol{\varepsilon}_h^q, \mathbf{d})_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^q \cdot \mathbf{n}, P\Theta \rangle_{\partial \mathcal{T}_h} + \langle \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u), \Pi \Theta \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

The following step consists of subtracting this equation from (5.5) to obtain

$$\|\varepsilon_h^u\|_\Omega^2 = \underbrace{(\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \Pi \mathbf{d})_{\mathcal{T}_h} - (\kappa^{-1} \boldsymbol{\varepsilon}_h^q, \mathbf{d})_{\mathcal{T}_h}}_{C_h^{(1)}} + \underbrace{\langle \widehat{\varepsilon}_h^u, \mathbf{d} \cdot \mathbf{n} \rangle_\Gamma - \langle \boldsymbol{\varepsilon}_h^q \cdot \mathbf{n} + \tau(\varepsilon_h^u - \widehat{\varepsilon}_h^u), P\Theta \rangle_{\partial \mathcal{T}_h}}_{C_h^{(2)}}. \quad (5.6)$$

*Bound for  $C_h^{(1)}$ .* Since  $\kappa^{-1} \mathbf{d} = -\nabla \Theta$ ,

$$\begin{aligned} C_h^{(1)} &= (\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \Pi \mathbf{d} - \mathbf{d})_{\mathcal{T}_h} + (\Pi \mathbf{q} - \mathbf{q}, -\kappa^{-1} \mathbf{d})_{\mathcal{T}_h} \\ &= (\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \Pi \mathbf{d} - \mathbf{d})_{\mathcal{T}_h} + (\Pi \mathbf{q} - \mathbf{q}, \nabla \Theta - \mathbf{P}_{k-1} \nabla \Theta)_{\mathcal{T}_h}, \end{aligned}$$

where  $\mathbf{P}_{k-1}$  is the local orthogonal projection on the spaces  $\mathcal{P}_{k-1}(K)$  if  $k \geq 1$  and  $\mathbf{P}_{-1} = 0$ . Note that the inclusion of  $\mathbf{P}_{k-1} \nabla \Theta$  is possible because of the definition of the HDG projection (4.1). Therefore

$$\begin{aligned} |C_h^{(1)}| &\leq Ch \|\mathbf{q} - \mathbf{q}_h\|_\Omega (|\mathbf{d}|_{H^1(\Omega)} + |\Theta|_{H^2(\Omega)}) + Ch^{\min\{k,1\}} \|\Pi \mathbf{q} - \mathbf{q}\|_\Omega |\Theta|_{H^2(\Omega)} \\ &\leq Ch^{\min\{k,1\}} \|\varepsilon_h^u\|_\Omega \left( \|\mathbf{q} - \mathbf{q}_h\|_\Omega + \|\Pi \mathbf{q} - \mathbf{q}\|_\Omega \right), \end{aligned} \quad (5.7)$$

by regularity (5.3) and [7, Theorem 2.2].

*Bound for  $C_h^{(2)}$ .* Let now  $\Theta_h$  be the Clément approximation [5] on a  $\mathbb{P}_1$  conforming finite element space on a triangulation  $\tilde{\mathcal{T}}_h$  whose restriction to the boundary is  $\Gamma_h$ . We then use (4.1c) to write

$$C_h^{(2)} = \langle \widehat{\varepsilon}_h^u, \widehat{\pi}_d \rangle_\Gamma - \langle \mathbf{q} \cdot \mathbf{n} - \widehat{\mathbf{q}}_h \cdot \mathbf{n}, \Theta_h \rangle_\Gamma + \langle \mathbf{q} \cdot \mathbf{n} - \widehat{\mathbf{q}}_h \cdot \mathbf{n}, \Theta_h - P\Theta \rangle_\Gamma, \quad (5.8)$$

where  $\widehat{\pi}_d := \Pi \mathbf{d} \cdot \mathbf{n} + \tau(\Pi\Theta - P\Theta)$ . Testing the error equation (4.9d) with  $\widehat{\pi}_d$ , and using (5.2), it follows that

$$\begin{aligned} \langle \widehat{\varepsilon}_h^u, \widehat{\pi}_d \rangle_\Gamma &= \langle (\tfrac{1}{2}\mathcal{I} + \mathcal{K})(\varphi - \varphi_h), \widehat{\pi}_d \rangle_\Gamma \\ &= \langle (\tfrac{1}{2}\mathcal{I} + \mathcal{K})(\varphi - \varphi_h), \widehat{\pi}_d - \mathbf{d} \cdot \mathbf{n} \rangle_\Gamma + \langle \mathcal{W}(\varphi - \varphi_h), \Theta \rangle_\Gamma \\ &= \langle (\tfrac{1}{2}\mathcal{I} + \mathcal{K})(\varphi - \varphi_h), \widehat{\pi}_d - \mathbf{d} \cdot \mathbf{n} \rangle_\Gamma + \langle \mathcal{W}(\varphi - \varphi_h), \Theta - \Theta_h \rangle_\Gamma \\ &\quad + \langle \mathbf{q} \cdot \mathbf{n} - \widehat{\mathbf{q}}_h \cdot \mathbf{n}, \Theta_h \rangle_\Gamma, \end{aligned} \quad (5.9)$$

where we have used that

$$\langle \mathbf{q} \cdot \mathbf{n} - \widehat{\mathbf{q}}_h \cdot \mathbf{n}, \phi \rangle_\Gamma + \langle \mathcal{W}(\varphi - \varphi_h), \phi \rangle_\Gamma = \omega(\varphi - \varphi_h, \phi) \quad \forall \phi \in Y_h,$$

which is just (4.9e) after using Lemmas 2.1 and 2.2. Inserting (5.9) in (5.8) and applying Propositions A.3 and A.4 we can bound

$$\begin{aligned} |C_h^{(2)}| &\leq C \|\varphi - \varphi_h\|_{H^{1/2}(\Gamma)} (\|\widehat{\pi}_d - \mathbf{d} \cdot \mathbf{n}\|_{H^{-1/2}(\Gamma)} + \|\Theta - \Theta_h\|_{H^{1/2}(\Gamma)}) \\ &\quad + \|\mathfrak{h}^{1/2}(\mathbf{q} \cdot \mathbf{n} - \widehat{\mathbf{q}}_h \cdot \mathbf{n})\|_\Gamma (\|\mathfrak{h}^{-1/2}(P\Theta - \Theta)\|_\Gamma + \|\mathfrak{h}^{-1/2}(\Theta - \Theta_h)\|_\Gamma). \end{aligned} \quad (5.10)$$

We now just need to bound all the terms in the right hand side of (5.10). A duality argument (Aubin-Nitsche trick), Lemma 4.1, and the regularity assumption (5.3) show that

$$\begin{aligned} \|\widehat{\pi}_d - \mathbf{d} \cdot \mathbf{n}\|_{H^{-1/2}(\Gamma)} &= \|P(\mathbf{d} \cdot \mathbf{n}) - \mathbf{d} \cdot \mathbf{n}\|_{H^{-1/2}(\Gamma)} \leq Ch_\Gamma^{1/2} \|\widehat{\pi}_d - \mathbf{d} \cdot \mathbf{n}\|_\Gamma \\ &\leq Ch \|\nabla \mathbf{d}\|_\Omega \leq Ch \|\varepsilon_h^u\|_\Omega. \end{aligned} \quad (5.11)$$

Well-known properties of the Clément interpolant (see in particular [8, Proposition 5.2]) and (5.3) prove also that

$$\|\Theta - \Theta_h\|_{H^{1/2}(\Gamma)} + \|\mathfrak{h}^{-1/2}(\Theta - \Theta_h)\|_\Gamma \leq Ch |\Theta|_{H^2(\Omega)} \leq Ch \|\varepsilon_h^u\|_\Omega. \quad (5.12)$$

Finally, we just overestimate

$$\|\mathfrak{h}^{-1/2}(P\Theta - \Theta)\|_\Gamma \leq \|\mathfrak{h}^{-1/2}(P_k\Theta - \Theta)\|_\Gamma \leq Ch^{\min\{1,k\}} |\Theta|_{H^2(\Omega)} \leq Ch^{\min\{1,k\}} \|\varepsilon_h^u\|_\Omega, \quad (5.13)$$

after applying a discrete trace inequality and (5.3). Bringing the bounds (5.11), (5.12) and (5.13) to (5.10) and using Corollary 4.4, we obtain the bound

$$|C_h^{(2)}| \leq Ch^{\min\{1,k\}} \|\varepsilon_h^u\|_\Omega (\text{App}_h^g + \|\varphi - \varphi_h\|_{H^{1/2}(\Gamma)} + \|P_\Gamma \varphi - \varphi_h\|_{H^{1/2}(\Gamma)}) \quad (5.14)$$

The bound (5.4) is now a direct consequence of (5.6), (5.7), (5.14) and Theorem 4.3.  $\square$

Note that in absence of any kind of additional regularity hypothesis, the estimate of Proposition 5.1 can be easily repeated without the additional  $h^{\min\{k,1\}}$  that provides superconvergence to the method. Also, as a consequence of Proposition 5.1, the local postprocessing technique of [7, Section 5] can be applied here providing a  $\mathcal{O}(h^{k+2})$  approximation of  $u$ .

**Corollary 5.2.** *In the hypotheses of Proposition 5.1, for  $k \geq 1$ ,*

$$\|\widehat{\varepsilon}_h^u\|_h := \left( \sum_{K \in \mathcal{T}_h} h_K \|\widehat{\varepsilon}_h^u\|_{\partial K}^2 \right)^{1/2} \leq Ch \left( \text{App}_h^q + \|P_\Gamma \varphi - \varphi\|_{H^{1/2}(\Gamma)} \right). \quad (5.15)$$

*Proof.* The argument to derive (5.15) from (5.4) can be taken verbatim from the proof of [7, Theorem 4.1].  $\square$

**Proposition 5.3.** *In the hypotheses of Theorem 4.3*

$$\|\varphi - \varphi_h\|_\Gamma \leq Ch^{1/2} \left( \text{App}_h^q + \|P_\Gamma \varphi - \varphi\|_{H^{1/2}(\Gamma)} \right) + \|\widehat{\varepsilon}_h^u\|_\Gamma.$$

*Proof.* The error equation (4.9d) is equivalent to writing  $\langle \widetilde{\mathcal{K}}(\varphi - \varphi_h), \widehat{v} \rangle_\Gamma = \langle \widehat{\varepsilon}_h^u, \widehat{v} \rangle_\Gamma$  for all  $\widehat{v} \in M_h$ . Therefore, by Lemmas 2.1 and 2.2 and Proposition A.4, we can bound

$$\begin{aligned} \|\varphi - \varphi_h\|_\Gamma &\leq C \sup_{0 \neq \phi \in H^0(\Gamma)} \frac{\langle \widetilde{\mathcal{K}}(\varphi - \varphi_h), \phi \rangle_\Gamma}{\|\phi\|_\Gamma} \\ &\leq C \sup_{0 \neq \phi \in H^0(\Gamma)} \frac{\langle \widetilde{\mathcal{K}}(\varphi - \varphi_h), \phi - P\phi \rangle_\Gamma}{\|\phi\|_\Gamma} + C \sup_{0 \neq \phi \in H^0(\Gamma)} \frac{\langle \widehat{\varepsilon}_h^u, P\phi \rangle_\Gamma}{\|\phi\|_\Gamma} \\ &\leq C \|\varphi - \varphi_h\|_{H^{1/2}(\Gamma)} \sup_{0 \neq \phi \in H^0(\Gamma)} \frac{\|\phi - P\phi\|_{H^{-1/2}(\Gamma)}}{\|\phi\|_\Gamma} + \|\widehat{\varepsilon}_h^u\|_\Gamma. \end{aligned}$$

An Aubin-Nitsche duality argument shows then that  $\|\phi - P\phi\|_{H^{-1/2}(\Gamma)} \leq h^{1/2} \|\phi\|_\Gamma$ . The proof then follows by Theorem 4.3.  $\square$

**Corollary 5.4.** *In the hypotheses of Proposition 5.1 and assuming that  $h \leq Ch_\Gamma$ , for  $k \geq 1$ ,*

$$\|\varphi - \varphi_h\|_\Gamma \leq Ch^{1/2} \left( \text{App}_h^q + \|P_\Gamma \varphi - \varphi\|_{H^{1/2}(\Gamma)} \right).$$

*Proof.* It is a direct consequence of Proposition 5.3, using Corollary 5.2, the estimate  $\|\widehat{\varepsilon}_h^u\|_\Gamma \leq \|\mathfrak{h}^{-1}\|_{L^\infty(\Gamma)}^{1/2} \|\widehat{\varepsilon}_h^u\|_h$ , and (4.11).  $\square$

## 6 Some additional considerations

**On the potential representation.** The main drawback of a boundary integral formulation based on a potential ansatz on a polyhedral boundary is the expected lack of regularity of the associated density. This makes that the hypotheses for regularity of the solution in Corollary 4.5 might not be realistic. Note, however, that the numerical experiments shown in Section 7, for which the density is unknown, show that the computation

of the interior unknowns and of the exterior solution are not affected for the foreseeable lack of regularity of the density in the corners of the domain. On the other hand, the analysis allows for using a very refined grid  $\Gamma_h$  near the corners. From the point of view of implementation, the case when  $\Gamma_h$  is a refinement of the grid  $\mathcal{T}_h$  restricted to  $\Gamma$ , avoids many of the complications of dealing with general non-matching grids.

**Direct Boundary Integral formulation.** A possible remedy for the above problem is the use of a direct formulation. For formulations of mixed type (and the one leading to the HDG method is one of such), this has been explained in great detail in [17]. The coupled formulation (2.7) has to be modified with the following arguments. First of all, we represent the exterior solution by

$$u_+ = \mathcal{D}\psi + \mathcal{S}(\mathbf{q} \cdot \mathbf{n} + \beta_1) \quad \psi \in H_0^{1/2}(\Gamma),$$

where  $\mathcal{S}$  is the single layer potential (A.3). Second, we consider the coupled problem

$$\kappa^{-1}\mathbf{q} + \nabla u^\circ = 0 \quad \text{in } \Omega, \quad (6.1a)$$

$$\operatorname{div} \mathbf{q} = f \quad \text{in } \Omega, \quad (6.1b)$$

$$u^\circ - \psi = \beta_0 \quad \text{on } \Gamma, \quad (6.1c)$$

$$-\langle \mathbf{q} \cdot \mathbf{n}, \tfrac{1}{2}\phi + \mathcal{K}\phi \rangle_\Gamma + \omega(\varphi, \phi) = \langle \beta_1, \tfrac{1}{2}\phi + \mathcal{K}\phi \rangle_\Gamma \quad \forall \phi \in H^{1/2}(\Gamma). \quad (6.1d)$$

Finally, the interior field  $u^\circ$  is corrected by adding a constant,  $u = u^\circ + c$ , where

$$c := \frac{1}{|\Gamma|} \left( \langle 1, \tfrac{1}{2}\psi + \mathcal{K}\psi \rangle_\Gamma + \langle \mathbf{q} \cdot \mathbf{n} + \beta_1, \eta \rangle_\Gamma \right) \quad \eta := \int_\Gamma \Phi(|\cdot - \mathbf{y}|) d\Gamma(\mathbf{y}),$$

and  $\Phi$  is the fundamental solution for the Laplacian (A.1). This gives the solution to (2.1). Without the correction only  $\mathbf{q}$  and  $u_+$  are correctly determined. If desired, it is possible to write  $\psi + c = \gamma u_+$  and use this as a way of obtaining an approximation of the exterior trace. Equations (2.9) can be easily modified to handle this reformulation. From the point of view of implementation, this requires a simple rearrangement of the matrices in (2.9). The fact that the integral operators appear also in the right hand side, in a term of the form  $\langle \beta_1, \tfrac{1}{2}\phi + \mathcal{K}\phi \rangle_\Gamma$ , with  $\phi \in Y_h$ , can be easily handled by preprojecting the data function  $\beta_1$  in the space  $M_h$  restricted to the boundary  $\Gamma$ . Note that the HDG-BEM discretization of (6.1) leads to a system whose matrix is the transpose of the one in Section 2. Therefore, the quadratic form is the same, and all the energy arguments can be applied. The part of the analysis related to duality arguments requires some additional work though.

**Non-symmetric RT-BEM.** A recent article [8] studies the symmetric coupling of HDG and BEM (using two integral equations) in parallel to the coupling of Raviart-Thomas mixed elements with BEM. The latter had appeared in the literature long ago [18, 2]. Among other things, [8] provided an analysis of superconvergence in  $L^2$  for the approximation of the variable  $u$ , and discussed the algorithmic advantages of hybridizing the RT method when coupled with BEM. Non-symmetric coupling of mixed elements with BEM has been proposed and studied in [17]. The analysis of this paper can be modified to include the methods proposed in [17], thus providing some improved estimates that the variational techniques in that paper did not show.



**On the diffusion parameter  $\kappa$ .** The final point for discussion is Hypothesis (3.1). First of all, let us mention that this inequality might not be sharp. We explore this in Section 7. However, there is evidence that some hypothesis like this is consubstantial to the mixed formulation that we are using for the HDG-BEM coupling. A similar result, with a lower bound for the minimum value of  $\kappa$  –i.e., an upper bound for  $\|\kappa^{-1}\|_{L^\infty}$ – had been noticed in the context of FEM-BEM formulations: see [22] and [13, Theorem 5.1]. It has been recently proved [20, Lemma 3.2] that there is actually a threshold for the minimum value of  $\kappa$  under which the formulation used in non-symmetric FEM-BEM loses its well-posedness. Nevertheless, if the threshold is crossed in points at a certain distance of the coupling boundary, a compactness argument can be invoked to show that the formulation and its Galerkin discretizations (for sufficiently refined grids) are well posed. While global compactness arguments (where lower order terms are added and subtracted to the equation) do not seem to be applicable for the current format of HDG analysis, it is possible that some equivalent ideas could be used to prove that the methods of this paper can be used by placing the boundary  $\Gamma$  sufficiently far from the places where  $\kappa$  is much larger than the exterior (unit) diffusivity.

## 7 Experiments

**Convergence and superconvergence.** For this example the domain is the square  $(0, 1) \times (0, 1)$ . The boundary mesh  $\Gamma_h$  is taken to be the restriction of  $\mathcal{T}_h$  to  $\Gamma$ . The coarsest grid contains only two triangles. Other grids are obtained by uniform refinement, reaching up to 2048 elements and 3136 edges, of which 128 are boundary edges. Note that for a polynomial degree  $k$ , the dimension of the system that is solved (with  $\hat{u}_h$  and  $\varphi_h$  as unknowns) is  $k + 1$  times the number of edges plus the number of boundary edges. We take the diffusion parameter  $\kappa(x, y) = 1 + x^2$  and  $u(x, y) = \exp(x + y)$  as exact solution in  $\Omega$ . The exterior solution is taken to be

$$u^+(\mathbf{x}) := -\frac{1}{2\pi} \log \frac{|\mathbf{x} - \mathbf{x}_1|}{|\mathbf{x} - \mathbf{x}_2|} \quad \mathbf{x}_1 = (0.3, 0.4), \quad \mathbf{x}_2 = (0.7, 0.6).$$

The density  $\varphi$  is not known. To check errors we approximate the exterior solution at the observation point  $\mathbf{x}_{\text{obs}} = (-0.1, 0.1)$ . Computation of the exterior solution is done using high order quadrature on the formula for the double layer potential. We tabulate and plot the following errors:

$$e_h^{\mathbf{q}} := \frac{\|\mathbf{q} - \mathbf{q}_h\|_\Omega}{\|\mathbf{q}\|_\Omega}, \quad e_h^{\hat{u}} := \frac{\|u - \hat{u}_h\|_h}{\|u\|_h}, \quad e_h^+ := \frac{|u_+(\mathbf{x}_{\text{obs}}) - u_{+,h}(\mathbf{x}_{\text{obs}})|}{|u_+(\mathbf{x}_{\text{obs}})|}$$

$$\varepsilon_h^u := \frac{\|\Pi u - u_h\|_\Omega}{\|u\|_\Omega}, \quad \varepsilon_h^{\hat{u}} := \frac{\|Pu - u_h\|_h}{\|u\|_h}.$$

The results are reported in Tables 1 to 3 for polynomials degree  $k = 0$  to  $k = 2$  respectively. Estimated convergence errors are computed using consecutive grids.

$e_h^{\mathbf{q}}$	ecr	$\widehat{e}_h^u$	ecr	$\varepsilon_h^u$	ecr	$e_h^+$	ecr	$\widehat{\varepsilon}_h^u$	ecr
3.9102(-1)	-	2.5573(-1)	-	3.1210(-2)	-	6.3929(0)	-	4.7747(-2)	-
2.2878(-1)	0.77	1.1908(-1)	1.10	1.4877(-2)	1.07	5.1969(-1)	3.62	2.6686(-2)	0.84
1.1867(-1)	0.95	5.5497(-2)	1.10	6.2955(-3)	1.24	3.5613(-1)	0.55	8.4902(-3)	1.65
6.0426(-2)	0.97	2.6790(-2)	1.05	3.5409(-3)	0.83	1.7215(-1)	1.05	3.8430(-3)	1.14
3.0435(-2)	0.99	1.3157(-2)	1.03	1.8998(-3)	0.90	8.6852(-2)	0.99	1.9418(-3)	0.98
1.5265(-2)	1.00	6.5192(-3)	1.01	9.8424(-4)	0.95	4.3598(-2)	0.99	9.9074(-4)	0.97

Table 1: Experiments for the lowest order ( $k = 0$ ) method. All errors behave like  $\mathcal{O}(h)$ .

$e_h^{\mathbf{q}}$	ecr	$\widehat{e}_h^u$	ecr	$\varepsilon_h^u$	ecr	$e_h^+$	ecr	$\widehat{\varepsilon}_h^u$	ecr
9.1892(-2)	-	3.2673(-2)	-	4.8524(-3)	-	9.3891(-2)	-	9.9900(-3)	-
2.5806(-2)	1.83	7.6059(-3)	2.10	7.6764(-4)	2.66	2.0486(-2)	2.20	1.6756(-3)	2.58
6.7443(-3)	1.94	1.7824(-3)	2.09	1.1487(-4)	2.74	1.2999(-3)	3.98	2.4053(-4)	2.80
1.7223(-3)	1.97	4.2887(-4)	2.06	1.5980(-5)	2.85	8.3227(-5)	3.97	3.2658(-5)	2.88
4.3549(-4)	1.98	1.0507(-4)	2.03	2.1137(-6)	2.92	1.1326(-5)	2.88	4.2930(-6)	2.93
1.0952(-4)	1.99	2.6000(-5)	2.01	2.7176(-7)	2.96	1.3269(-6)	3.09	5.5141(-7)	2.96

Table 2: Experiment for the case  $k = 1$ . Errors for  $\mathbf{q}$  and  $u$  on  $\partial\mathcal{T}_h$  behave like  $\mathcal{O}(h^2)$ . Comparison of  $u$  with respect to the projections (in elements and on their boundaries), as well as the exterior potential superconverge like  $\mathcal{O}(h^3)$ .

**Tests related to the diffusion parameter.** For this example, the domain is the rectangle  $(-3/2, 3/2) \times (-1, 1)$  and we use  $k = 0$ . We start with a fixed triangulation (produced with MATLAB's PDE Toolbox) with 936 elements. We take two different diffusion parameters: a constant value  $\kappa(\mathbf{x}) \equiv \kappa_{cons}$  and a piecewise constant function

$$\kappa(\mathbf{x}) := \begin{cases} \kappa_{int} & \text{in } (-3/4, 3/4) \times (-1/2, 1/2), \\ 1 & \text{otherwise.} \end{cases}$$

We note that the values  $\kappa_{cons} = 0$  and  $\kappa_{int} = 0$  make the problem degenerate. We also note that the jump in the discontinuous diffusion coefficient is not resolved by the triangulation, i.e., we do not choose a triangulation with edges on the jump of the coefficient. As a test that might allow us to understand the effect of the diffusion parameter on the coupled system, we compute the condition number for increasing values of  $\kappa_{cons}$  and  $\kappa_{int}$  and plot them together in Figure 1. It is clear from the results for constant diffusion that the condition (4.1) is too restrictive –see also the comments in Section 6 concerning the recent results on nonsymmetric BEM-FEM–, but that growth of this parameter increases the condition number of the system significantly. It is also clear that if the diffusion coefficient grows far from the boundary, the problem is much better conditioned and that, for this case, the growth of conditioning appears to be linear in this parameter, which agrees with the basic fact that the matrix is an affine function of  $\kappa^{-1}$  that seems not to degenerate as  $\kappa$  grows.

We then take several concrete values (three constant diffusion parameters and two piecewise constant, with the same notation as above), and plot the condition number for uniformly refined grids. Results are shown in Figure 2.

$e_h^{\mathbf{q}}$	ecr	$\widehat{e}_h^u$	ecr	$\varepsilon_h^u$	ecr	$e_h^+$	ecr	$\widehat{\varepsilon}_h^u$	ecr
1.2540(-2)	-	2.7770(-3)	-	5.9374(-4)	-	7.5351(-2)	-	9.3898(-4)	-
1.7437(-3)	2.85	3.2768(-4)	3.08	3.9253(-5)	3.92	1.6677(-2)	2.18	9.6755(-5)	3.28
2.3006(-4)	2.92	3.8082(-5)	3.11	2.7636(-6)	3.83	5.8133(-5)	8.16	7.6652(-6)	3.66
2.8039(-5)	3.04	4.5361(-6)	3.07	1.4710(-7)	4.23	1.4569(-6)	5.32	4.1376(-7)	4.21
3.5088(-6)	3.00	5.5514(-7)	3.03	9.0340(-9)	4.03	2.6695(-9)	9.09	2.5795(-8)	4.00
4.3945(-7)	3.00	6.8673(-8)	3.02	5.7394(-10)	3.98	9.5947(-10)	1.48	1.6331(-9)	3.98

Table 3: Experiment for the case  $k = 2$ . Errors for  $\mathbf{q}$  and  $u$  on  $\partial\mathcal{T}_h$  behave like  $\mathcal{O}(h^3)$ . Comparison of  $u$  with respect to the projections (in elements and on their boundaries), as well as the exterior potential superconverge like  $\mathcal{O}(h^4)$ . The exterior field behaves somewhat erratically, which might be due to unaccounted errors in BEM quadrature.

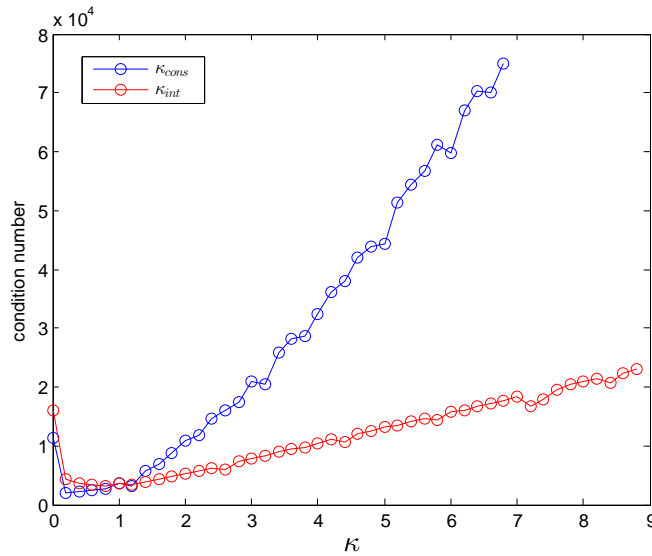


Figure 1: Condition number of the system matrix for growing values of the diffusion parameter in the entire domain ( $\kappa_{cons}$ ) and in an interior subdomain ( $\kappa_{int}$ ).

## A Exterior and transmission problems: a compendium

In this section we collect known results about layer potentials, as well as exterior and transmission problems associated to the Laplace equation. The results are straightforward consequences of results that are contained in [16, Chapters 6 & 8].

### A.1 Layer potentials

Let

$$\Phi(r) = \Phi_d(r) := \begin{cases} -1/(2\pi) \log r, & \text{when } d = 2, \\ 1/(4\pi r), & \text{when } d = 3, \end{cases} \quad (\text{A.1})$$

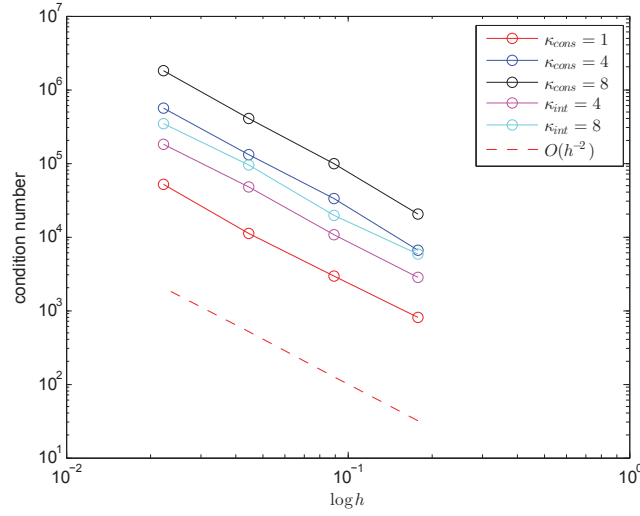


Figure 2: Condition number of the system matrix for different diffusion parameters and uniformly refined grids. All of them behave like  $\mathcal{O}(h^{-2})$ , as could be expected from a two dimensional elliptic problem.

be the fundamental solution of the Laplace equation. The double layer potential with density  $\varphi \in H^{1/2}(\Gamma)$  is:

$$\begin{aligned} (\mathcal{D}\varphi)(\mathbf{x}) &:= \int_{\Gamma} \nabla_{\mathbf{y}} \Phi(|\mathbf{x} - \mathbf{y}|) \cdot \mathbf{n}(\mathbf{y}) \varphi(\mathbf{y}) \, d\Gamma(\mathbf{y}) \\ &= \frac{1}{2(d-1)\pi} \int_{\Gamma} \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} \varphi(\mathbf{y}) \, d\Gamma(\mathbf{y}), \quad \forall \mathbf{x} \in \mathbb{R}^d \setminus \Gamma. \end{aligned} \quad (\text{A.2})$$

The single layer potential is defined with a duality product: for  $\lambda \in H^{-1/2}(\Gamma)$ , we define  $(\mathcal{S}\lambda)(\mathbf{x}) := \langle \lambda, \Phi(|\mathbf{x} - \cdot|) \rangle_{\Gamma}$ , on any  $\mathbf{x} \in \mathbb{R}^d \setminus \Gamma$ . When  $\lambda \in L^2(\Gamma)$ , this duality product can be written in integral form

$$(\mathcal{S}\lambda)(\mathbf{x}) = \int_{\Gamma} \Phi(|\mathbf{x} - \mathbf{y}|) \lambda(\mathbf{y}) \, d\Gamma(\mathbf{y}). \quad (\text{A.3})$$

**Proposition A.1.** *Let  $\varphi \in H^{1/2}(\Gamma)$  and  $u := \mathcal{D}\varphi$ . Then:*

$$u \in H^1(\Omega) \text{ and } u \in H^1(\Omega_+ \cap B(\mathbf{0}; R)) \quad \forall R, \quad (\text{A.4a})$$

$$\Delta u = 0 \text{ in } \mathbb{R}^d \setminus \Gamma, \quad (\text{A.4b})$$

$$u \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \Gamma) \text{ and } u = \mathcal{O}(r^{-d+1}) \text{ as } r = |\mathbf{x}| \rightarrow \infty, \quad (\text{A.4c})$$

$$\gamma^+ u - \gamma^- u = \varphi \text{ and } \partial_{\mathbf{n}}^+ u - \partial_{\mathbf{n}}^- u = 0 \text{ on } \Gamma. \quad (\text{A.4d})$$

Moreover,

$$\mathcal{D}1 = -\chi_{\Omega} = \begin{cases} -1 & \text{in } \Omega, \\ 0, & \text{in } \Omega_+. \end{cases} \quad (\text{A.5})$$

*Proof.* See [16, Theorem 6.11] and [16, Chapter 8].  $\square$

The fact that constant densities produce a vanishing exterior solution of the Laplace equation motivates the introduction of the space

$$H_0^{1/2}(\Gamma) := \{\varphi \in H^{1/2}(\Gamma) : \int_{\Gamma} \varphi = 0\}. \quad (\text{A.6})$$

## A.2 Exterior solutions of the Laplace equation

We consider functions  $u_+ : \Omega_+ \rightarrow \mathbb{R}$  such that

$$\Delta u_+ = 0 \text{ in } \Omega_+ \quad \text{and} \quad u_+ \in H^1(\Omega_+ \cap B(\mathbf{0}; R)) \quad \forall R. \quad (\text{A.7})$$

Since, by Weyl's lemma, locally integrable solutions of the Laplace equation are  $\mathcal{C}^\infty$ , it then makes sense to impose a strong radiation condition at infinity. The general asymptotic condition we will deal with has the form

$$u = c_\infty \Phi(r) + \mathcal{O}(r^{1-d}) \text{ as } r \rightarrow \infty, \text{ uniformly in all directions.} \quad (\text{A.8})$$

Note that this condition includes logarithmically growing solutions when  $d = 2$ , unless  $c_\infty = 0$ . The incoming flux on  $\Gamma$  is defined as

$$c_{\text{flux}} := -\langle \partial_{\mathbf{n}} u_+, 1 \rangle_{\Gamma}. \quad (\text{A.9})$$

**Proposition A.2.** *Let  $u$  satisfy (A.7) and (A.8). Then:*

(a)  $u_+$  admits the representation formula

$$u_+ = \mathcal{D}\gamma u_+ - \mathcal{S}\partial_{\mathbf{n}} u_+ = \mathcal{D}(\gamma u_+ + c) - \mathcal{S}\partial_{\mathbf{n}} u_+ \quad \forall c \in \mathbb{P}_0(\Gamma). \quad (\text{A.10})$$

(b)  $c_\infty = c_{\text{flux}}$  and therefore, a necessary and sufficient condition for  $u$  to be decaying at infinity in the two dimensional case is  $c_{\text{flux}} = 0$ .

(c) For every  $\mathbf{x}_0 \in \Omega$ , there exists a unique  $\phi \in H_0^{1/2}(\Gamma)$  such that

$$u_+ = c_{\text{flux}} \Phi(|\cdot - \mathbf{x}_0|) + \mathcal{D}\phi. \quad (\text{A.11})$$

Therefore,  $u_+$  can be represented as a double layer potential if and only if  $u_+ = \mathcal{O}(r^{-d+1})$  at infinity.

*Proof.* Part (a) is a well-known representation formula [16, Theorem 7.15]. The inclusion of any additive constant in the input of  $\mathcal{D}$  follows from (A.5). Part (b) is straightforward using the strong integral form of the potentials. Part (c) can be easily proved by considering  $u_+ - c_{\text{flux}} \Phi(|\cdot - \mathbf{x}_0|)$  as the solution of an exterior Neumann problem and using a double layer potential representation [16, Theorem 8.19–8.21].  $\square$

### A.3 Integral operators

Because of the transmission conditions satisfied by the double layer potential (A.4d), we can define the operators

$$\mathcal{W}\varphi := -\partial_{\mathbf{n}}^{\pm}\mathcal{D}\varphi, \quad \mathcal{K}\varphi := \frac{1}{2}(\gamma^{+}\mathcal{D}\varphi + \gamma^{-}\mathcal{D}\varphi). \quad (\text{A.12})$$

Note that the conditions (A.4d) and the definition of  $\mathcal{K}$  imply that

$$\gamma^{\pm}\mathcal{D}\varphi = \pm\frac{1}{2}\varphi + \mathcal{K}\varphi. \quad (\text{A.13})$$

When  $\Gamma$  is a polyhedral ( $d = 3$ ) or polygonal ( $d = 2$ ) boundary, the integral expression of the operator  $\mathcal{K}$

$$(\mathcal{K}\varphi)(\mathbf{x}) = \frac{1}{2(d-1)\pi} \int_{\Gamma} \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} \varphi(\mathbf{y}) \, d\Gamma(\mathbf{y}) \quad (\text{A.14})$$

is valid on all points  $\mathbf{x} \in \Gamma$  that do not lie on edges. In particular (A.14) holds almost everywhere on  $\Gamma$ . Also, if  $\Gamma$  is a polyhedral boundary ( $d = 3$ ) and  $\varphi, \phi \in L^{\infty}(\Gamma)$  are such that  $\nabla_{\Gamma}\varphi, \nabla_{\Gamma}\phi \in L^{\infty}(\Gamma)^d$  (here  $\nabla_{\Gamma}$  is the tangential gradient), we have an integral form for the bilinear form associated to  $\mathcal{W}$ :

$$\langle \mathcal{W}\varphi, \phi \rangle = \int_{\Gamma} \int_{\Gamma} (\mathbf{n}(\mathbf{x}) \times \nabla_{\Gamma}\varphi(\mathbf{x})) \cdot (\mathbf{n}(\mathbf{y}) \times \nabla_{\Gamma}\phi(\mathbf{y})) \Phi(|\mathbf{x} - \mathbf{y}|) \, d\Gamma(\mathbf{x})d\Gamma(\mathbf{y}). \quad (\text{A.15})$$

In the two dimensional case, the bilinear form is

$$\langle \mathcal{W}\varphi, \phi \rangle = \int_{\Gamma} \int_{\Gamma} \partial_{\tau}\varphi(\mathbf{x})\partial_{\tau}\phi(\mathbf{y}) \Phi(|\mathbf{x} - \mathbf{y}|) \, d\Gamma(\mathbf{x})d\Gamma(\mathbf{y}), \quad (\text{A.16})$$

where  $\partial_{\tau}$  is the tangential derivative on  $\Gamma$ .

**Proposition A.3** (Properties of  $\mathcal{W}$ ). *The operator  $\mathcal{W}$  is bounded  $H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ . Its kernel is the set of constant functions. The bilinear form  $\omega : H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \rightarrow \mathbb{R}$ ,*

$$\omega(\varphi, \phi) := \langle \mathcal{W}\varphi, \phi \rangle_{\Gamma} + \int_{\Gamma} \varphi \int_{\Gamma} \phi$$

*is bounded, symmetric, and coercive. Also  $\omega(\varphi, \varphi) = \langle \mathcal{W}\varphi, \varphi \rangle_{\Gamma}$  for all  $\varphi \in H_0^{1/2}(\Gamma)$  and*

$$\langle \mathcal{W}\varphi, \varphi \rangle_{\Gamma} = (\nabla u_{\star}, \nabla u_{\star})_{\mathbb{R}^d \setminus \Gamma}, \quad \text{where } u_{\star} = \mathcal{D}\varphi, \quad \varphi \in H^{1/2}(\Gamma).$$

*Proof.* See [16, Theorems 8.20 & 8.21]. □

**Proposition A.4** (Properties of  $\mathcal{K}$ ). *The operator  $\mathcal{K}$  is bounded  $H^{1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$  and  $H^0(\Gamma) \rightarrow H^0(\Gamma)$ . Moreover,*

$$\langle \partial_{\mathbf{n}}^{+}\mathcal{S}\lambda, \varphi \rangle_{\Gamma} = \langle \lambda, -\frac{1}{2}\varphi + \mathcal{K}\varphi \rangle_{\Gamma} \quad \forall \lambda \in H^{-1/2}(\Gamma), \quad \varphi \in H^{1/2}(\Gamma).$$

*Finally*

$$\|\xi - \frac{1}{|\Gamma|} \int_{\Gamma} \xi\|_{\Gamma} \leq C \sup_{0 \neq \phi \in L^2(\Gamma)} \frac{\langle \frac{1}{2}\xi + \mathcal{K}\xi, \phi \rangle_{\Gamma}}{\|\phi\|_{\Gamma}} \quad \forall \xi \in H^0(\Gamma). \quad (\text{A.17})$$

*Proof.* Boundedness in  $H^{1/2}(\Gamma)$  and the transposition property follow from the variational theory of layer potentials: see [16, Theorems 6.11 & 6.17]. For the  $H^0(\Gamma)$  boundedness techniques of harmonic analysis are needed [24]. The bound (A.17) follows from the fact that  $\frac{1}{2}\mathcal{I} + \mathcal{K}$  is Fredholm of index zero and its kernel is the set of constant functions. □

## A.4 Transmission problems

Let  $0 \leq \kappa \in L^\infty(\Omega)$  be such that  $\kappa^{-1} \in L^\infty(\Omega)$ . The data of the transmission problem are  $f \in L^2(\Omega)$ ,  $\beta_0 \in H^{1/2}(\Gamma)$ ,  $\beta_1 \in H^{-1/2}(\Gamma)$ . We look for  $u : \Omega \rightarrow \mathbb{R}$  and  $u_+ : \Omega_+ \rightarrow \mathbb{R}$  satisfying: the exterior Laplace equation (A.7) with radiation condition (A.8), the interior elliptic equation

$$u \in H^1(\Omega) \quad -\nabla \cdot (\kappa \nabla u) = f \text{ in } \Omega, \quad (\text{A.18})$$

and the transmission conditions

$$\gamma u = \gamma u_+ + \beta_0 \text{ on } \Gamma \quad \text{and} \quad \kappa \nabla u \cdot \mathbf{n} = \partial_{\mathbf{n}} u_+ + \beta_1 \text{ on } \Gamma. \quad (\text{A.19})$$

Because of (A.18)-(A.19), it is easy to prove that  $c_{\text{flux}} = c_{\text{data}}$ , where

$$c_{\text{data}} := \int_{\Omega} f + \langle \beta_1, 1 \rangle_{\Gamma}. \quad (\text{A.20})$$

**Proposition A.5** (Transmission problem). *The transmission problem looking for  $u, u_+$  and  $c_\infty$  satisfying (A.7), (A.8), (A.18), and (A.19), is uniquely solvable.*

- (a) *When  $d = 2$ , the solution is decaying ( $c_\infty = 0$  in (A.8)) if and only if  $c_{\text{data}} = 0$ .*
- (b) *When  $d = 3$ , the solution satisfies  $u_+ = \mathcal{O}(r^{-2})$  as  $r \rightarrow \infty$  if and only if  $c_{\text{data}} = 0$ . Finally, if  $c_{\text{data}} \neq 0$  and  $\mathbf{x}_0 \in \Omega$ , we can write*

$$u_+ = c_{\text{data}} \Phi(|\cdot - \mathbf{x}_0|) + u_+^d,$$

where  $u_+^d$  satisfies (A.7), and  $u_+^d = \mathcal{O}(r^{-2})$  at infinity. The transmission conditions can then be written

$$\gamma u = \gamma u_+^d + \tilde{\beta}_0 \text{ on } \Gamma \quad \text{and} \quad \kappa \nabla u \cdot \mathbf{n} = \partial_{\mathbf{n}} u_+^d + \tilde{\beta}_1 \text{ on } \Gamma,$$

where

$$\tilde{\beta}_0 := \beta_0 + c_{\text{data}} \Phi(|\cdot - \mathbf{x}_0|), \quad \tilde{\beta}_1 := \beta_1 + c_{\text{data}} \nabla \Phi(|\cdot - \mathbf{x}_0|) \cdot \mathbf{n},$$

and therefore

$$\int_{\Omega} f + \langle \tilde{\beta}_1, 1 \rangle_{\Gamma} = 0.$$

*Proof.* Existence and uniqueness of solution of (A.7), (A.8), (A.18), and (A.19) can be easily proved using a symmetric boundary-field formulation: see [12, Section 1.5] for the general methodology and the needed background results.  $\square$

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